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EXPLOSIVE INSTABILITY IN NONLINEAR WAVES IN MEDIA WITH NEGATIVE VISCOSITY

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The propagation of a wave of a finite amplitude in a medium with a nonlinearity of the second degree and negative viscosity, is examined. It is shown that in a finite time singularities appear in the solution. The exact solution of the Cauchy problem is given for a specific case. Recently the effects of negative viscosity which cause an increase in the energy of the wave motion have been studied intensively in electrodynamics, plasma physics, the Earth's atmosphere, in the theory of the circulation of the oceans and of flow in open channels [1–4]. Wave amplification caused by an energy transfer from turbulent to regular motions, is possible in any medium having space-time fluctuations, provided the correlation time is sufficiently small [5, 6]. As the wave amplitude increases, nonlinear effects become important; they have been taken into account in cases where the interaction of a finite number of harmonics [2, 4] and the structure of steady motions have been examined [3].

It is shown in this paper that in a medium with negative viscosity and a second degree dynamic nonlinearity, a solution of the Cauchy problem for an arbitrary "good" form of the initial perturbation, exists over a finite time interval. An example of such a solution is given.

The equation describing approximately the propagation of a wave having a small but finite amplitude in a medium with negative viscosity, has the form

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \delta \frac{\delta^2 u}{\partial x^2} = 0 \tag{1}$$

Here u is a certain physical variable describing the state of the medium (the value αu has the dimension of velocity), α and δ are constants. Equation (1) can be obtained from the Navier-Stokes equation by a recurrence procedure, taking into account a small negative viscosity [7]. We note that this equation can be reduced to the Burgers equation [8] by the change of variables $t \to -t$, $x \to -x$, $u \to -u$ (1) . Consequently, knowing the shock wave, the solution of Eq. (1) can be used to construct the field for $t < 0$ in a medium with ordinary viscosity.

For (1) we consider the Cauchy problem with the initial condition

$$u(x, 0) = U(x) \tag{2}$$

where we assume that the function U is sufficiently smooth. In the case of $\delta < 0$ (which corresponds to an ordinary viscosity) the character of the process is determined by the Reynolds number [8]. If $\mathrm{Re} \ll 1$, the effect of nonlinearity is unessential and the wave is rapidly attenuated. For $\mathrm{Re} \gg 1$ the wave is initially distorted as an ordinary one, the steepness of the forward slope increases and is limited by dissipation. The structure of the shock front is well described by the stationary solution of the Burgers equation. If $\delta > 0$ and $\mathrm{Re} \gg 1$, the wave first distorts as an ordinary one but afterwards the negative viscosity becomes essential and, unlike in the case of ordinary viscosity, affects the stability.

Let us discuss first the structure of the stationary shock waves in such a medium. From (1) we easily obtain the stationary solution

$$u_0 = V\left[\mathrm{th}\,\frac{\alpha V z}{2\delta} + 1\right], \qquad z = x - \alpha V t \tag{3}$$

depending on the parameter V. As is evident from (3), such a solution cannot approximate the front of an arbitrary perturbation for $\mathrm{Re} \gg 1$, as it does in the case of positive viscosity. It can be shown that the stationary wave (3) is unstable because small perturbations tend to move from the "discontinuity". The time characterizing the development of instability can be evaluated by linearizing (1) at the stationary solution (3) and representing the perturbations in the form

$$\delta u = u - u_0 = \sum_m \psi_m(z)\, e^{\lambda_m t}\, \mathrm{sch}\,\frac{\alpha V z}{2\delta} \tag{4}$$

where ψ is an eigenfunction of the Schrödinger equation

$$\delta \frac{d^2\psi}{dz^2} + [E - U_*(z)]\,\psi = 0 \tag{5}$$

$$E = \lambda - \frac{\alpha^2 V^2}{4\delta}, \qquad U_* = -\frac{\alpha^2 V^2}{2\delta}\,\mathrm{sch}^2\,\frac{\alpha V z}{2\delta}$$

The solution of (5) is well known [9], and as a result we find that the spectrum of Eq. (5) consists of two discrete levels $\lambda_0 = 0$ and $\lambda_1 = 3\alpha^2 V^2/16\delta$ and a continuous region for $\lambda \geqslant \alpha^2 V^2/4\delta$. The perturbation at $\lambda_0 = 0$ corresponds to a shifting of the stationary wave as a whole (cf. [10]) and the remaining ones correspond to the deformation of the wave contour. Hence, the "life time" of the stationary wave does not exceed $16\delta / 3\alpha^2 V^2$, which is of the order of the duration of the stationary gradient. Therefore the long-duration existence of stationary shock waves is impossible in a medium with negative viscosity.

It is convenient to examine the evolution of nonstationary perturbations using the linear equation

$$\frac{\partial\theta}{\partial t} + \delta \frac{\partial^2\theta}{\partial x^2} = 0 \tag{6}$$

obtained from (1) by substituting [8]

$$u = \frac{2\delta}{\alpha\theta}\frac{\partial\theta}{\partial x}, \qquad \theta = \mathrm{const}\,\exp\int_0^x \frac{\alpha u(t, x')}{2\delta}\,dx' \tag{7}$$

First of all we note that the Cauchy solution for Eq. (6) is equivalent to the solution of the inverse problem of the theory of heat conduction. By virtue of the linearity in (6) we represent the solution in the form of the Fourier-Stieltjes integral

$$\theta(x, t) = \int_{-\infty}^{\infty} \theta(k) \exp[\delta k^2 t + ikx]\,dk \tag{8}$$

where $\theta(k)$ is the spectrum of the function $\theta(x, 0)$. Hence, if for $k \to \infty$ the value $\theta(k)\exp(\delta k^2 \tau)$ tends to zero sufficiently rapidly for any τ, then the function $\theta(x, t)$ is bounded at any instant. For large t and limited x the integral in (8) can be computed by the method of steepest descent. As a result, we obtain (k_0 (t) is the point of the descent)

$$\theta(x, t) \sim e^{\gamma(t)} \cos[k_0(t)x - \nu(t)] \tag{9}$$
$$2\delta k_0 t = -\frac{d}{dk_0}\ln|\theta(k_0)|, \quad \gamma(t) = \delta k_0^2 t + \ln|\theta(k_0)|$$

where ν is proportional to the argument $\theta(k_0)$. From (9) we find the asymptotic expression for u

$$u(x, t) \sim -\frac{2\delta k_0}{\alpha}\operatorname{tg}(k_0 x - \nu) \tag{10}$$

Hence, the field $u(x, t)$ at the points $k_0 x = \nu \pm \pi/2$ becomes infinite for a finite t. This time ("explosion time") corresponds to the instant of the first crossing of the level $\theta = 0$ by the function $\theta(x, t)$ which initially, according to (7), was of constant sign and, of course, belonged to the class of functions with a fairly smooth spectrum. If for large k the function $\theta(k) \sim \exp(-\delta k^2 \tau)$, then the solution is limited only for $t < \tau$, and the time of explosion coincides with τ. Finally, if $\theta(k)$ descreases more slowly than $\exp(-\delta k^2 \tau)$, then the integral in (8) diverges for an arbitrarily small t and the analysis of the solution is impossible by means of (8).

Thus, the field $u(t, x)$ becomes infinite in a finite time. This result was obtained previously for a finite number of interacting sinusoidal waves [2, 4]. We note that in the linear theory, a case is possible in which the field remains constant at any instant (cf.(9)); for this reason the spectrum of the initial perturbations must be restricted (a sufficiently rapid decrease in the region of large wave numbers). It is evident that small distortions of the wave shape (variations in the high frequency part of the spectrum) can result in an unlimited field, due to their amplification at high frequencies. The incorrectness of the inverse problem in the theory of the heat conduction, is related to this fact [11]. In a nonlinear medium an unlimited field appears for an arbitrary "good" form of the initial perturbation. Physically this is related to the chain reaction which high frequency harmonics generated, due to the nonlinearity, are amplified more strongly the higher the frequency. We note that this also proves the incorrectness of the inverse problem in the theory of ordinary waves in a medium with a small positive viscosity and it was impossible to determine the wave shape in the class of the bounded functions for $t \to -\infty$.

The character of the development of instability and the explosion time can be examined by means of the moments of the field. For example, let $U(x)$ be

$$\int_{-\infty}^{\infty} U(x)\,dx = 0$$

Introducing the moments $P_m(t)$ and making use of (6), we find for this case

$$P_0(t) = P_0 = \text{const}, \quad P_1(t) = P_1(0) - 2\delta P_0 t \tag{11}$$
$$P_m(t) = \int_{-\infty}^{\infty} x^{2m}[\theta(x, t) - \theta(\pm\infty, 0)]\,dx$$

At $t = 0$ all moments have the same sign, e.g. positive, but in the course of time some of them, namely the odd ones, become negative. This means that a region with a negative value of field appears in the initially positive impulse. By virtue of (7), the discontinuities in the function $u(x, t)$ occur with the change of the sign in the function $\theta(x, t)$. We note that the boundedness of the moments P_m near T_* indicates the integrable character of the appearing singularities. An estimate of the instant of explosion T_* follows from (11)

$$T_* \leqslant T_1 = P_1(0) / 2\delta P_0$$

For example, for a Gaussian impulse the explosion time corresponds to the time of its turning into a delta function. The explosion time can be also found for all self-similar solutions obtained from the known quantities [12] by changing $\delta \to -\delta$ and $t \to \tau - t$ (τ is the explosion time). By means of the moments it is also possible to evaluate the explosion time for other types of initial conditions for the function $U(x)$ either of constant sign or periodic.

Finally, we give an exact solution of the problem for the function of the form

$$U(x) = \frac{U_0 \sin kx}{1 + {}^1\!/_2 \mathrm{Re} \cos kx} \quad \left(\mathrm{Re} = \frac{\alpha U_0}{k\delta} < 2\right)$$

the solution of which has the form

$$u(x, t) = \frac{U_0 \exp(\delta k^2 t) \sin kx}{1 + {}^1\!/_2 \mathrm{Re} \exp(\delta k^2 t) \cos kx}$$

We note that in the example considered here, the function θ is expressed by the sum of only two terms and the function $u(x, t)$ is represented at any instant by an infinite Fourier series

$$\theta(x, t) = 1 + {}^1\!/_2 \mathrm{Re} \exp(\delta k^2 t) \cos kx$$

$$u(x, t) = \frac{4U_0}{\mathrm{Re}} \sum_{n=1}^{\infty} (-1)^{n+1} \left(2 \exp \frac{(-\delta k^2 t)}{\mathrm{Re}}\right)^n \times$$

$$[1 - \sqrt{1 - (\mathrm{Re}/2)^2 \exp(2\delta k^2 t)}]^n \sin nkx$$

The solution remains continuous for $t < T_* = -(\delta k^2)^{-1} \ln \mathrm{Re}/2$; near T_* the wave spectrum increases rapidly in the region of large wave numbers.

Thus, a wave of arbitrary amplitude in a medium with a quadratic nonlinearity and a negative viscosity, remains bounded only over a finite time interval. We note that Eq. (1) is derived for small but finite amplitudes (see [7]), so that in the region of explosion it is necessary to take into account factors causing absorption either at high frequencies or at high amplitudes. Stabilization of the nonlinear process is possible at some definite level.

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REFERENCES

1. Starr, V., Physics of Phenomena with Negative Viscosity. Moscow, "Mir", 1971.
2. Mashkovich, S. A., and Veil', I. G., Large scale atmospheric processes with a "negative viscosity". Meteorologiia i Gidrologiia, № 8, 1970.
3. Johnson, R. S., Shallow water waves on a viscous fluid – the undular bore. Phys. Fluids, Vol. 15, № 10, 1972.
4. Rabinovich, M. I. and Reutov, V. P., Interaction of parametrically related waves in nonequilibrium media. Izv. vuzov. Radiofizika, Vol. 16, № 6, 1973.
5. Gavrilenko, V. G. and Dorfman, Ia. M., On the theory of wave dissipation in a medium with space-time fluctuations. Izv. vuzov. Radiofizika, Vol. 15, № 2, 1972.

6. McGorman, R. E. and Mysak, L. A., Internal waves in a randomly stratified fluid. Geophys. Fluid Dynam., Vol. 4, № 3, 1973.
7. Ostrovskii, L. A. and Pelinovskii, E. N., Approximate equations for waves in media with small nonlinearity and dispersion. PMM Vol. 38, № 1, 1974.
8. Karpman, V. I., Nonlinear Waves in Dispersive Media. Moscow, "Nauka", 1973.
9. Landau, L. D. and Lifshits, E. M., Quantum Mechanics. English translation, Pergamon Press, Book № 09101, 1965.
10. Barenblatt, G. I. and Zel'dovich, Ia. B., On the stability of flame propagation. PMM Vol. 21, № 6, 1957.
11. Lattes, R. and Lions, J. L., The Method of Quasi-reversibility: Applications to Partial Differential Equations. New York, American Elsevier Publ. Co., 1969.
12. Rudenko, O. V. and Soluian, S. I., Some nonstationary problems of the theory of finite amplitude waves in dissipative media. Dokl. Akad. Nauk SSSR, Vol. 190, № 4, 1970.

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ON THE STABILITY OF WEAKLY INHOMOGENEOUS STATES WITH A SMALL ADDITION OF WHITE NOISE

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Using the concepts developed in [1] we investigate, in the presence of certain restrictions, the stability of a weakly inhomogeneous state parametrically perturbed by a small random addition of white noise. We show that when the characteristic wavelength is arbitrarily small as compared with the distance over which it varies substantially, then the mechanism of formation of the eigenfunctions responsible for the stability of the state is analogous to the mechanism given in [1]. In the present case it is not the boundaries that act as reflectors, as in [1], but the points at which the condition of existence of the global eigenfunction for the homogeneous problem holds. We obtain the criterion of stability of the state in question and discuss the problem of application of the results obtained to the case in which the ratio of the characteristic wavelength to the distance over which it varies substantially, cannot be taken as arbitrarily small.

1. The statement of the problem is analogous to that given in [1]. We consider the following homogeneous problem:

$$\sum_{ikm}\left[a_{ikml}(\varepsilon x) + h d_{ikml}(\varepsilon x)\, F\left(\frac{x}{\delta}\right)\right] D_{ik}\Psi_m = 0, \qquad D_{ik} = \frac{\partial^i}{(\partial x)^i}\frac{\partial^k}{(\partial t)^k} \quad (1.1)$$

$$\left(\sum_{ikm} f_{ikml} D_{ik}\Psi_m\right)_{x=0}, \qquad l = 1, 2, \ldots, n$$

$$\left(\sum_{ikm} f_{ikml} D_{ik}\Psi_m\right)_{x=L/\varepsilon}, \qquad l = n+1, \ldots, N, \;\; L \sim 1$$